SOME PROBLEMS OF THE THEORY OF MASS AND HEAT TRANSPORT

A. V. Lykov

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Let us first examine some urgent questions of the theory of heat conduction. The coefficient of heat conduction Λ is a tensor for anisotropic media. Hence, it is interesting to obtain a number of solutions of heat conduction problems for anisotropic solids. The coefficient of heat conduction of some fluids is also a tensor at sufficiently high speeds. In this case the coefficient of heat conduction depends on the strain rate tensor [1].

However, the differential equations of the heat conduction of materials with variable memory [2] are of greatest interest.

1. Heat Conduction of Materials with Memory

The specific heat flux is defined in [3] by the relationship

$$\mathbf{q}(\tau) = -\int_0^\infty k(\theta)\,\mathbf{g}(\tau-\theta)\,d\theta, \tag{1.1}$$

where $\mathbf{g} \equiv \operatorname{grad} T$ is the temperature gradient. Therefore, the specific heat flux is independent of the magnitude of the temperature gradient at a given time. Moreover, (1.1) is valid for isotropic materials.

If the temperature gradient $\mathbf{g}$ is independent of the time θ, then letting λ denote the quantity

$$\lambda = \int_0^\infty k(\theta)\,d\theta, \tag{1.2}$$

we obtain the classical formula for the Fourier heat conduction law $\mathbf{q} = -\lambda \operatorname{grad} T$, where λ is the coefficient of heat conduction in the equilibrium state.

If it is assumed that

$$k(\theta) = \frac{\lambda}{\tau_r} \exp\left(-\frac{\theta}{\tau_r}\right), \tag{1.3}$$

where τ_r is the Maxwell—Cattaneo relaxation time [4, 5], then we obtain an expression for the heat flux taking account of the finite rate of heat propagation in the form

$$\mathbf{q} = -\lambda \operatorname{grad} T - \tau_r \dot{\mathbf{q}}. \tag{1.4}$$

The relaxation time τ_r is inversely proportional to the square of the finite rate of heat propagation w_q and is directly proportional to the coefficient of temperature heat conduction a ($\tau_r = a/w_q^2$).

If the linearized governing equation for the heat flux $\mathbf{q}$ and the internal energy e is used in the form [3]

$$\mathbf{q} = -\int_0^\infty k(\theta)\,\mathbf{g}(\tau-\theta)\,d\theta = \int_0^\infty k'(\theta)\,\bar{\mathbf{g}}^\tau(\theta)\,d\theta, \tag{1.5}$$

$$e = e_0 + CT - \int_0^\infty \beta'(\theta)\,\bar{T}^\tau(\theta)\,d\theta, \tag{1.6}$$

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where

$$\frac{d}{d\theta}\bar{\mathbf{g}}^{\tau}(\theta)=\mathbf{g}^{\tau}(\theta)=\mathbf{g}(\tau-\theta), \tag{1.7}$$

C is the volume specific heat of the solid, $\beta(\theta)$ is the internal energy relaxation function, then we obtain the following relationships for the time derivatives of the heat flux $\dot{\mathbf{q}}$ and internal energy $\dot{e}$:

$$\dot{\mathbf{q}}=-k(0)\,\mathbf{g}-\int_0^{\infty}k'(\theta)\,\mathbf{g}(\tau-\theta)\,d\theta, \tag{1.8}$$

$$\dot{e}=C\dot{T}+\beta(0)\,T+\int_0^{\infty}\beta'(\theta)\,T(\tau-\theta)\,d\theta. \tag{1.9}$$

Then if the energy conservation equation is used, we obtain the following generalized linear equation of heat conduction:

$$C\frac{\partial^2 T(\mathbf{x},\tau)}{\partial\tau^2}+\beta(0)\frac{\partial T(\mathbf{x},\tau)}{\partial\tau}+\int_0^{\infty}\beta'(\theta)\frac{\partial T(\mathbf{x},\tau-\theta)}{\partial\tau}\,d\theta$$

$$=k(0)\nabla^2 T(\mathbf{x},\tau)+\int_0^{\infty}k'(\theta)\nabla^2 T(\mathbf{x},\tau-\theta)\,d\theta+\dot{Q}(\mathbf{x},\tau), \tag{1.10}$$

where $Q(\mathbf{x},\tau)$ is the external heat source. Equation (1.10) differs from the hyperbolic differential equation of heat conduction in the presence of additional integral terms taking account of the relaxation functions of the heat conduction and the internal energy. The relations (1.8)-(1.9) show that the relaxation times of the thermal stress and the internal energy are different.

A linearized governing equation for the heat flux in the form

$$\mathbf{q}(\tau)=-\lambda(0)\nabla T(\tau)-\int_0^{\infty}\lambda'(\theta)\nabla T(\tau-\theta)\,d\theta, \tag{1.11}$$

which takes account of the dependence of the heat flux on the temperature gradient at a given time, is used in [6] for an isotropic medium. If it is assumed that $\lambda'(\theta)=0$, then we obtain the classical Fourier equation of heat conduction. For $\lambda(0)=0$ we obtain a relationship analogous to (1.1). The quantity $\lambda(0)$ is called the instantaneous coefficient of heat conduction. In this case, the heat conduction differential equation will be

$$C(0)\frac{\partial T(\mathbf{x},\tau)}{\partial\tau}+\int_0^{\infty}\alpha'(\theta)\frac{\partial T(\mathbf{x},\tau-\theta)}{\partial\tau}\,d\theta=\lambda(0)\nabla^2 T(\mathbf{x},\tau)$$

$$+\int_0^{\infty}\lambda'(\theta)\nabla^2 T(\mathbf{x},\tau-\theta)\,d\theta+\dot{Q}(\mathbf{x},\tau). \tag{1.12}$$

The linearized governing equation for the internal energy

$$e=e_0+C(0)\,T+\int_0^{\infty}\alpha'(\theta)\,T(\tau-\theta)\,d\theta, \tag{1.13}$$

was used in deriving (1.12), where $\alpha(\theta)$ is the relaxation internal energy function, and C(0) is the instantaneous volume specific heat.

The differential equation (1.12) has been solved for a half-space when the kernels of the integral relations $\alpha(\theta)$ and $\lambda(\theta)$ are power-law or exponential functions of the time θ. The presence of integral relations in the heat conduction equation (1.12) does not introduce major difficulties in its solution by the method of the Laplace integral transform since the integration in these relations is with respect to the time between the limits 0 and ∞ [7]. Temperature waves in materials with memory are of special interest since they have the singularity that their propagation rates and damping coefficients differ from the analogous relationships in the classical theory of heat conduction.

2. Heat Conduction and Diffusion with Discontinuous Transport Coefficients

The simplest heat conduction problems with moving boundaries are the problems about the freezing of moist ground or the hardening of a fixed fluid (the Stefan problem). In these problems, the phase transformation surface, the crystallization surface, say, moves within the body. Many papers [8-9] are devoted to the solution of this kind of problem. These problems can be reduced to problems with fixed boundaries but with the introduction of discontinuous functions for the analytical description of the transport coefficients and the effective specific heat. For example, the one-dimensional Stefan problem

$$C_i(T_i)\frac{\partial T_i}{\partial \tau} = \frac{\partial}{\partial x}\left[\lambda_i(T)\frac{\partial T_i}{\partial x}\right]; \quad \begin{matrix} i=1, \ 0<x<\xi(\tau); \\ i=2, \ \xi<x<l \end{matrix} \tag{2.1}$$

with the boundary conditions

$$T_1(x, 0) = \varphi_1(x) \leqslant T_f; \quad T_2(x, 0) = \varphi_2(x) \geqslant T_f; \tag{2.2}$$

$$T_1(0, \tau) = f_1(\tau) < T_f; \quad T_2(l, \tau) = f_2(\tau) > T_f; \tag{2.3}$$

$$T_1(\xi, \tau) = T_2(\xi, \tau) = T_f = \text{const}; \tag{2.4}$$

$$\lambda_2(T)\frac{\partial T_2}{\partial x} - \lambda_1(T)\frac{\partial T_1}{\partial x} = r\frac{\partial \xi}{\partial \tau} \tag{2.5}$$

is reduced in [10] to the heat conduction problem with fixed boundaries

$$C_e(T)\frac{\partial T}{\partial \tau} = \frac{\partial}{\partial x}\left[\lambda(T)\frac{\partial T}{\partial x}\right], \tag{2.6}$$

where

$$C_e(T) = C(T) + r\delta(T - T_f); \tag{2.7}$$

$$C(T) = \begin{cases} C_1(T), & T<T_f; \\ C_2(T), & T>T_f; \end{cases} \quad \lambda(T) = \begin{cases} \lambda_1(T), & T<T_f; \\ \lambda_2(T), & T>T_f, \end{cases} \tag{2.8}$$

and $\delta(T - T_f)$ is the delta function.

Therefore, the heat conduction coefficient $\lambda(T)$ and the effective specific heat $C_e(T)$ vary in jumps upon achievement of a temperature equal to the phase transformation temperature ($T = T_f$). Integrating (2.6) with respect to x between ($\xi - \varepsilon$) and ($\xi + \varepsilon$), where ε is an infinitesimal, we obtain the relation

$$\lim_{\varepsilon \to 0}\left\{\int_{\xi-\varepsilon}^{\xi+\varepsilon} C(T)\frac{\partial T}{\partial \tau}\,dx + \int_{T(\xi-\varepsilon,\tau)}^{T(\xi-\varepsilon,\tau)} r\delta(T - T_f)\frac{\partial x}{\partial \tau}\,dT\right\} = r\frac{d\xi}{d\tau}, \tag{2.9}$$

which permits complete solution of the problem.

Using discontinuous functions (Dirac delta functions, the Heaviside unit step H-function) permits a new mathematical formulation of the problem of heating porous bodies for a cavity of the phase transformation surface. In this case the ordinary differential equations of heat and mass transport for the drying of moist capillary-porous bodies can be used

$$\frac{\partial T}{\partial \tau} = a\nabla^2 T + \frac{\varepsilon r}{c}\frac{\partial u}{\partial \tau}, \tag{2.10}$$

$$\frac{\partial u}{\partial \tau} = a_{m2}\nabla^2 u + a_{m2}\delta_2\nabla^2 T + \varepsilon\frac{\partial u}{\partial \tau}, \tag{2.11}$$

where ε, the coefficient of phase transformation ($0 < \varepsilon < 1$), is described by the following discontinuous function

$$\varepsilon = H(u) - H(u - u_f); \tag{2.12}$$

H(u) is the Heaviside unit function:

$$H(u) = \begin{cases} 1, & u \geqslant 0; \\ 0, & u < 0; \end{cases} \tag{2.13}$$

u_f is the moisture content corresponding to the evaporation surface (see Fig. 1a). If the coefficient ε is a continuous function of the moisture content, then in some cases it can be represented as (see Fig. 1b)

$$\varepsilon = H(u) - (1 - \varepsilon_0)H(u - u_f), \tag{2.14}$$

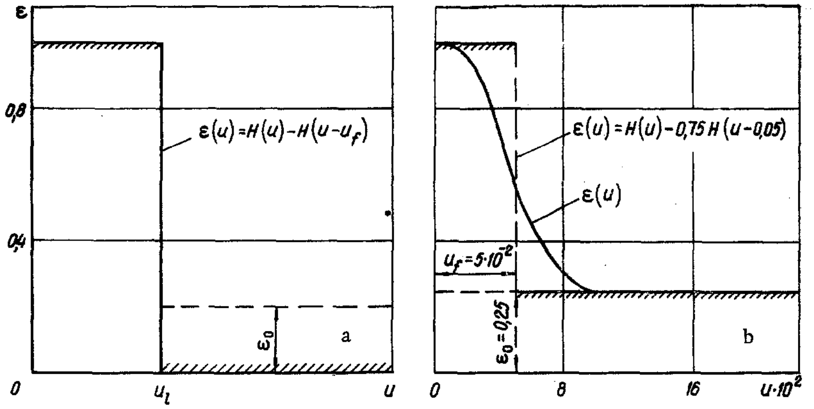

Fig. 1. Dependence of the phase transformation criterion ε; a) on the mass content u; b) experimental curve of ε(u) for quartz sand and its approximation by a discontinuous function. Moisture content u, kg/kg ($u_l \equiv u_f$).

where ε_0 is the value of the coefficient ε(u) for a moisture content $u \geq u_f$.

The analytical solution of the differential equations (2.10)-(2.11) for a discontinuous value of the coefficient ε(u) given by (2.12), is fraught with difficulties. However, the problem is solved completely by numerical methods.

The problem can be simplified for the drying of some capillary-porous bodies by taking the phase transformation coefficient dependent on the coordinates and time. For example, for the one-dimensional problem we can take

$$\varepsilon(x, \tau) = H(x) - H(x - b\tau), \tag{2.15}$$

where b is the rate of deepening of the evaporation surface ($b = d\xi/d\tau$), which is a constant for a number of materials [11]. The heat gradient coefficient δ depends on the moisture content; this dependence δ(u) is a sinusoid. In a first approximation it can be represented as the following relationship (see Fig. 2):

$$\delta(u)/\delta_m = \frac{1}{u_m}\{u[H(u)-H(u-u_m)]+(2u_m-u)[H(u-u_m)-H(u-2u_m)]\}. \tag{2.16}$$

The maximum value of the heat gradient coefficient δ_m, corresponding to the maximum heat content u_m (swelling moisture content), enters as constants in (2.16).

Therefore, an urgent question in the theories of heat conduction and heat and mass transport is the solution of the transport differential equations when the transport properties and the thermodynamic characteristics of the materials vary by a jump, i.e., are described by discontinuous functions. Such a description more truly reflects the physical transport mechanism and does not require finding empirical formulas for the transport coefficients and the thermodynamic properties as a function of the temperature and the mass content.

3. Heat and Mass Transport in Capillary-Porous Bodies

A great deal of attention has recently been paid to questions of heat and mass transport in capillary-porous bodies in connection with their use as wicks in heat pipes. If the gradients of the concentration $\nabla u(\nabla u \equiv \nabla\psi_1)$, the temperature $\nabla T(\nabla T \equiv \nabla\psi_2)$ and the total pressure (vapor and inert gas pressure) $\nabla P(\nabla P \equiv \nabla\psi_3)$ are taken as thermodynamic motive forces for the transport of the heat carrier (fluid and vapor), then the transport differential equations are

$$\frac{\partial\psi_i}{\partial\tau} = \sum_{j=1}^{3} K_{ij}\nabla^2\psi_j, \tag{3.1}$$

for a number of simplifications (constancy of the transport coefficients and of the thermodynamic characteristics), and without taking account of the convective heat transport, where the kinetic coefficients K_{ij} ($i, j = 1, 2, 3$) are determined in terms of the heat and mass transport coefficients and the thermodynamic characteristics [12].

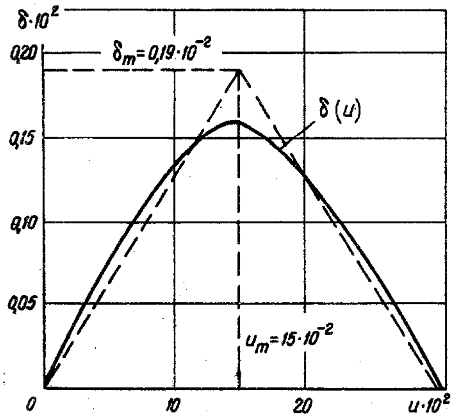

Fig. 2. Dependence of the heat gradient coefficient δ on the moisture content u for a colloidal capillary-porous body (clay at t = 35°C) and approximation of the curve δ(u) by a discontinuous function.

For a number of capillary-porous bodies the specific mass content u_1 of the vapor and u_3 of the inert gas is negligibly small as compared with the specific mass content of the fluid u_2, i.e., the fluid mass content equals the total mass content $u(u_2 \approx \sum_i u_i = u)$.

In this case, the system of three transport differential equations (3.1) can be reduced to a system of two transport differential equations by eliminating the quantity $\nabla^2 \psi_3$ by means of the relation for the vapor source I_{12}:

$$I_{12} = -I_{21} = a_{m1}\rho_0 (\nabla^2 u + \delta_1 \nabla^2 T) = k_p \nabla^2 P. \tag{3.2}$$

It follows from (3.2) that a total pressure gradient originates because of intense evaporation and the presence of resistance of the vapor and gas to the filtration stream through the capillary-porous body. Therefore, the system of heat and mass transport differential equations taking account of convective heat transport becomes

$$\frac{\partial T}{\partial \tau} = K_{11} \nabla^2 T + K_{12} \nabla^2 u - \sum_i (c_{pi}/c\rho_0)\, \mathbf{j}_i \cdot \nabla T, \tag{3.3}$$

$$\frac{\partial u}{\partial \tau} = K_{21} \nabla^2 T + K_{22} \nabla^2 u, \tag{3.4}$$

where the coefficients $K_{ij} (i, j = 1, 2)$ are determined in terms of the transport coefficients (see [12]).

If the assumption is made that the vapor pressure within the body is a single-valued function of the temperature $p_1 = f(T)$ then for an insignificant partial pressure gradient of the inert gas within the capillaries of the body $(\nabla p_3 \ll \nabla p_1)$ and independence of $\partial P/\partial T$ from the coordinates, (3.3)-(3.4) simplify and reduce to the equations

$$\frac{\partial T}{\partial \tau} = a_e \nabla^2 T + \gamma (\nabla T)^2, \tag{3.5}$$

$$\frac{\partial u}{\partial \tau} = a_{m2} \nabla^2 u + a_{m2} \delta_e \nabla^2 T, \tag{3.6}$$

where a_e, δ_e are the equivalent coefficients to a and δ, equal to

$$a_e = a + \frac{r k_p}{c\rho_0} \left(\frac{\partial P}{\partial T} \right); \quad \delta_e = \delta_2 + \frac{k_p}{\rho_0 a_{m2}} \left(\frac{\partial P}{\partial T} \right); \tag{3.7}$$

γ is the coefficient of convective transport

$$\gamma = \frac{c_{p1} k_p}{c\rho_0} \left(\frac{\partial P}{\partial T} \right). \tag{3.8}$$

We hence neglect the convective heat transport by the fluid. If convective heat transport is neglected ($\gamma = 0$), then (3.5) goes over into the classical Fourier equation of heat conduction. The solutions of the system (3.5)-(3.6) are well known.

For a stationary process taking account of the change in the transport coefficient and in convective heat transport, we will have

$$\operatorname{div}(\lambda\nabla T) + r\operatorname{div}(k_p\nabla P) + c_1k_p\nabla P\cdot\nabla T$$

$$+ c_2a_{m2}(\nabla u + \delta_2\nabla T)\cdot\nabla T = 0, \tag{3.9}$$

$$\operatorname{div}[a_{m2}(\nabla u + \delta_2\nabla T)] + \operatorname{div}(k_p\nabla P) = 0. \tag{3.10}$$

Hence, taking account of vapor transport is carried out on the basis of the filtration law ($j_1 = -k_p \nabla P$). If the radius of the fluid meniscus in the capillaries is greater than 10^{-5} cm, then it is possible to take the relation

$$\nabla P = \left(\frac{\partial P}{\partial T}\right)\nabla T, \tag{3.11}$$

i.e., we neglect the influence of curvature of the meniscus surface on the vapor pressure.

The presence of the pressure gradient produces a condition for the ejection of fine fluid particles from the porous body into the surrounding medium. These fluid particles are incident on the vapor layer located near the body surface, where their evaporation indeed occurs. The transport coefficients (λ, a_{m2}, $a_{m2}\delta$) in the transport equations presented above should be determined experimentally; they depend on the structure of the porous body. It is hence assumed that the temperature of the capillary walls equals the temperature of the fluid or vapor in the capillary. Such an assumption is justified for small transport intensities. If the transport intensity is great, then such an assumption will be incorrect. Moreover, the velocity of heat carrier motion will vary across the capillary section. In this case the mean temperature and velocity of heat carrier motion are introduced. It is expedient to use a different description of the heat and mass transport (separately for the fluid and vapor) with the insertion of the structural–mechanical characteristics for such a model. The simplest such characteristic is the porosity Π.

Let us denote the mean value of any quantity ψ (with respect to the fluid-filled volume of the pore V_s) by $\bar{\psi}$, i.e.,

$$\bar{\psi} = \frac{1}{V}\int\limits_{V_s}\psi dV. \tag{3.12}$$

Then the mean temperature $\bar{T}$ and the mean linear velocity of fluid motion $\bar{v}$ will equal respectively

$$\bar{T} = \frac{1}{1-\Pi}\bar{T}_0 = \frac{1}{\Pi}\bar{T}_2;\quad \bar{\mathbf{v}}_2 = \frac{\mathbf{j}_2}{\rho_2\Pi}. \tag{3.13}$$

In the case of fluid motion within a porous body (fluid evaporation occurs only on the body surface), we will have

$$[c_2\rho_2\Pi + c_0\rho_0(1-\Pi)]\frac{\partial\bar{T}}{\partial\tau} + c_2\rho_2\operatorname{div}(\Pi\bar{\mathbf{v}}_2\bar{T})$$

$$= \lambda_2\operatorname{div}[\nabla(\Pi\bar{T})] + \lambda_0\operatorname{div}[\nabla(1-\Pi)\bar{T}] + \operatorname{div}\mathbf{q}^*, \tag{3.14}$$

where the heat flux vector **q*** is determined by the relationship [13]

$$\mathbf{q}^* = c_2\rho_2[\bar{\mathbf{v}}_2\bar{T}_2 - \overline{\mathbf{v}_2T_2}] + (\lambda_2 - \lambda_0)\frac{1}{V}\int\limits_{A_i}\bar{T}\mathbf{n}^1 dA. \tag{3.15}$$

Integration occurs along the fluid surface A_i in contact with the capillary walls of the bodies, $\mathbf{n}^1$ is the unit normal to the surface A bounding the volume V.

It follows from (3.14) that in contrast to the Fourier–Kirchhoff equation, heat transport occurs not only by means of heat conduction

$$\mathbf{q} = -[\lambda_2\Pi - \lambda_0(1-\Pi)]\operatorname{grad}\bar{T}, \tag{3.16}$$

but in addition also because of the contact exchange between the body skeleton and the fluid (the integral term in (3.15) and by means of "turbulent" heat exchange (the expression in square brackets in (3.15)).

Indeed, the heat flux

$$c_2\rho_2[\bar{\mathbf{v}}_2\bar{T}_2 - \overline{\mathbf{v}_2T_2}] \tag{3.17}$$

is the analog of the turbulent heat flux q_{tur} in a moving fluid

$$\mathbf{q}_{tur} = c_2\rho_2(\bar{T}\bar{\mathbf{v}} - \overline{T\mathbf{v}}) = c_2\rho_2\overline{T'\mathbf{v}'}. \tag{3.18}$$

Therefore, additional heat transport due to the complex mechanism of fluid motion exists in a capillary-porous body.

Let us examine this question in greater detail. In the majority of cases turbulent heat transport is considered proportional to the temperature gradient

$$\mathbf{q}_{tur} = c_p\rho\,(\bar{T}\,\bar{\mathbf{v}} - \overline{T\mathbf{v}}) = \lambda_{tur}\nabla\bar{T}, \tag{3.19}$$

where λ_{tur} is the coefficient of turbulent heat conduction ($\lambda_{tur} = c_p\rho a_{tur}$) for a fluid stream ($c_p = c_2$; $\rho_2 = \rho$; $\lambda_2 = \lambda$). However, the turbulent transfer can be described differently. Let us see the Fourier—Kirchhoff heat transport equation for a turbulent incompressible fluid flow (ρ = const); we will have*

$$c_p\rho\left(\frac{\partial T}{\partial\tau} + \mathrm{div}\,\mathbf{v}T\right) = \mathrm{div}\,(\lambda\nabla T). \tag{3.20}$$

If the average with respect to time is taken

$$\bar{T}(\tau) = \frac{1}{\Delta\tau}\int\limits_{\tau}^{\tau+\Delta\tau} T(\tau)\,d\tau, \quad \bar{\mathbf{v}}(\tau) = \frac{1}{\Delta\tau}\int\limits_{\tau}^{\tau+\Delta\tau} \mathbf{v}(\tau)\,d\tau, \tag{3.21}$$

then $\bar{T}$, $\bar{\mathbf{v}}$ will be the mean values of the temperature T, and the velocity **v** in the time interval $\Delta\tau$. We therefore obtain

$$c_p\rho\,\frac{\partial\bar{T}}{\partial\tau} + c_p\rho\,\mathrm{div}\,(\overline{\mathbf{v}T}) = \mathrm{div}\,(\lambda\nabla\bar{T}). \tag{3.22}$$

Later we usually proceed thus:

$$\overline{\mathbf{v}T} = \bar{\mathbf{v}}\,\bar{T} - \bar{\mathbf{v}}\,\bar{T} + \overline{\mathbf{v}T} = \bar{\mathbf{v}}\,\bar{T} - (\bar{\mathbf{v}}\,\bar{T} - \overline{\mathbf{v}T}). \tag{3.23}$$

We consequently obtain the Fourier—Kirchhoff equation with the additional term $\mathrm{div}(\bar{\mathbf{v}}\bar{T} - \overline{\mathbf{v}T})$ which is called the divergence of the turbulent heat flux $\mathrm{div}\,\mathbf{q}_{tur}$, i.e.,

$$c_p\rho\left(\frac{\partial\bar{T}}{\partial\tau} + \bar{\mathbf{v}}\cdot\nabla\bar{T}\right) = \mathrm{div}\,(\lambda\nabla\bar{T}) + \mathrm{div}\,(\lambda_{tur}\nabla\bar{T}). \tag{3.24}$$

However, another path is also possible. The quantity $\bar{\mathbf{v}}\mathrm{T}$ is a vector, and it can be compared with the vector $\bar{\mathbf{v}}\bar{\mathrm{T}}$ by using the second rank tensor **B**:

$$\overline{\mathbf{v}T} = \mathbf{B}\cdot\bar{\mathbf{v}}\,\bar{T}. \tag{3.25}$$

Then we will have

$$c_p\rho\,\frac{\partial\bar{T}}{\partial\tau} + (c_p\rho\nabla\bar{T})\cdot(\bar{\mathbf{v}}\cdot\mathbf{B})$$

$$+ c_p\rho\bar{T}\,\mathrm{tr}\,(\nabla\bar{\mathbf{v}}\,\mathbf{B}) + c_p\rho\bar{T}\,\bar{\mathbf{v}}\cdot\mathrm{div}\,\mathbf{B} = \mathrm{div}\,(\lambda\nabla\bar{T}) \tag{3.26}$$

in place of (3.24). Since the fluid is incompressible, then $\mathrm{div}\,\bar{\mathbf{v}} = 0$.

Equation (3.26) is essentially different from (3.24) since a tensor, the coefficient of turbulent transport **B** enters instead of the coefficient of turbulent heat conduction.

From our viewpoint, the passage from the vector $\overline{\mathbf{v}\mathrm{T}}$ to the vector $\bar{\mathbf{v}}\bar{\mathrm{T}}$ by means of (3.25) is more correct than using the relations (3.19) and (3.23).

Analogous conversions can be made for (3.17) by using (3.25). However, in this case the tensor transform B which sets up the connection between $\overline{\mathbf{v}\mathrm{T}}$ and $\bar{\mathbf{v}}\bar{\mathrm{T}}$ is unique but not certainly linear. The transformation matrix is determined from an equation which can be solved by using the boundary conditions on the surface A_i. Therefore, no additional hypotheses with the introduction of the coefficient of turbulent heat conduction is required in the case of capillary-porous bodies.

The integral relation in (3.15) determines the additional heat flux on the interface between the fluid and the capillary walls.

An analogous situation holds for the filtering motion of a fluid within a porous body.

An additional force with which the fluid stream acts on the porous structure besides the hydrostatic pressure and other external forces originates as a fluid moves within a porous body. The magnitude of this force f_G computed per unit volume equals [13]

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*To shorten the derivation we neglect the Rayleigh dissipation function.

$$f_G = \frac{1}{V}\int_{A_i} [\mathbf{P} - p\mathbf{I}]\cdot \mathbf{n}^1 dA = R\bar{\mathbf{v}}, \tag{3.27}$$

where R is the drag coefficient which is a function of the fluid velocity averaged locally with respect to the volume V_S. It is hence assumed that the porous body is a disoriented porous structure. Then the differential equation of filtration fluid transport will be

$$\nabla p - \eta \operatorname{div}(\nabla\bar{\mathbf{v}}) + R\bar{\mathbf{v}} = 0, \tag{3.28}$$

which differs from the customary Darcy filtration equation by the presence of the additional term $R\bar{v}$. The Poiseuille formula for laminar fluid motion in capillaries is obtained as a particular case (R = 0).

4. About Momentum Transport

The fundamental four mechanics and thermodynamics fields in the theory of nonlinear thermomechanics of materials with memory are described by the governing functionals $\Psi_{(j)}$ (j = 1, 2, 3, 4). The governing functionals for a simple material are [2]

$$\begin{matrix} u \\ s \\ \mathbf{q} \\ \mathbf{\Pi} \end{matrix} = \left\{ \Psi_{(j)}(\mathbf{G}^{\tau},\ T^{\tau},\ \mathbf{g}(\tau)) \quad j = 1,\ 2,\ 3,\ 4, \right. \tag{4.1}$$

where

$$\mathbf{G}^{\tau}(\theta) = \mathbf{G}(\tau - \theta);\quad T^{\tau}(\theta) = T(\tau - \theta),$$

G is the strain gradient, u is the internal energy, s is the entropy, and Π is the stress tensor.

The governing functionals should satisfy a number of nonlinear thermomechanics principles, including the principle of the erasing memory (postulate of regularity) and the Clasius–Duhem inequality.

Coleman [14] defines memory as the recollection of the mean value of the quantity multiplied by a coefficient of forgetfulness. The coefficient of forgetfulness diminishes the contribution of the preceding events even more, the greater the time interval since the beginning of the event. In conformity with the principles of fading memory, the stress will differ slightly from the equilibrium stress if the deformation had been slight in the recent past.

The fading memory characterizes the material in the sense that the configurations which the body possessed in the recent past exert stronger influence on these values of the stress and free energy than do configurations of the remote past.

If the governing functional is continuous and differentiable, then it can be approximated by a simpler functional.

As has been shown above (§ 1), the simplest kind of memory is taking account of the Maxwell finite rate of propagation of the appropriate stress.

According to (1.9), the heat conduction $k(\theta)$ relaxes according to an exponential law, which results in a correction to the Fourier heat conduction law in the form of taking account of the finite heat propagation velocity (see (1.4)).

Analogously, the Newton momentum transport law (impulse) is based on the assumption that the velocity of momentum propagation w_σ is an infinite quantity ($w_\sigma = \infty$).

Taking account of the finite momentum propagation velocity

$$w_\sigma = \sqrt{\frac{G}{\rho}} = \sqrt{\frac{\eta}{\rho\tau_{r\sigma}}} = \sqrt{\frac{\nu}{\tau_{r\sigma}}}, \tag{4.2}$$

where G is the elastic shear modulus, and $\tau_{r\sigma}$ is the period of shear stress relaxation, results in a generalized formula for the internal friction stress

$$\sigma_{ij} = \eta\dot{\varepsilon}_{ij} - \tau_{r\sigma}\dot{\sigma}_{ij}, \tag{4.3}$$

where $\dot{\varepsilon}_{ij}$ is the strain rate tensor ($\dot{\varepsilon}_{ij} = d\varepsilon_{ij}/d\tau$).

The dot above the symbol denotes the time derivative. From (4.3) we obtain the classical formula of Newton's law for internal friction if we set $\tau_{r\sigma} = 0$ ($w_\sigma = \infty$), and the classical Hooke formula for the shear strain of an ideal elastic body if we assume that the coefficient of viscosity tends to infinity ($\eta \to \infty$, $w_\sigma \neq \infty$):

$$\sigma = G\varepsilon = \rho w_\sigma^2 \varepsilon. \tag{4.4}$$

It is completely natural that the turbulent friction stress tensor also relaxes during a finite time interval $\tau_{r\sigma}^{tur}$. Hence, in the general case the Reynolds turbulent stress tensor will be described by the generalized formula

$$\sigma^{tur} = \rho\overline{v'v'} + \tau_{r\sigma}^{tur}\dot{\sigma}^{tur}. \tag{4.5}$$

Taking account of these generalizations, the Navier–Stokes momentum transport differential equation will have a different form. To estimate the contribution introduced by these generalizations, the magnitude of the friction stress relaxation time $\tau_{r\sigma}$ or the rate of momentum propagation (w_σ) and the turbulent stress relaxation time $\tau_{r\sigma}^{tur}$ must be known. These quantities can only be determined experimentally.

However, it can be assumed from an analysis of the physical mechanism of molecular and turbulent momentum exchange that the turbulent stress relaxation time is considerably greater than the molecular fraction stress relaxation ($\tau_{r\sigma}^{tur} \gg \tau_{r\sigma}$).

It should be noted that turbulent momentum transport cannot be described by ordinary traditional-means when the turbulent stress tensor

$$\sigma_{tur} = \rho\overline{v'v'} = \rho\overline{vv} - \rho\bar{v}\,\bar{v} \tag{4.6}$$

is considered a function of the velocity gradient (for an incompressible fluid), and on the basis of another conversion of the diad $\rho\overline{vv}$ into the diad $\rho\bar{v}\bar{v}$ analogous to the relation (3.25). In this case we obtain a momentum transport equation in another form, different from the Navier–Stokes equation. It must be noted that the turbulent stress tensor is ordinarily considered symmetric.

In his original paper [15], Reynolds differentiated σ_{ij}^{tur} and σ_{ji}^{tur}, i.e., considered the turbulent stress tensor nonsymmetric. The extension to the case of an asymmetric tensor σ^{tur} is given in [16-17].

The analysis presented above shows convincingly the fruitfulness and urgency of using the methods of nonlinear thermomechanics of continuous media in the theory of momentum, energy, and mass transport phenomena.

The development of a nonlinear theory of heat and mass exchange is the primary problem in this modern domain of scientific knowledge.

NOTATION

g	is the temperature gradient;
T	is the temperature;
$k(\theta)$	is the heat flux relaxation function;
w_q	is the velocity of heat propagation;
a	is the coefficient of temperature conduction ($a = \lambda/c\rho$);
e	is the internal energy;
e_0	is the constant internal energy;
C	is the volume specific heat ($C = c\rho$);
c	is the specific heat;
$k(0)$	is the instantaneous heat conduction;
$c(0)$	is the instantaneous specific heat;
$\mathbf{x}$	is the position vector in Cartesian space;
x	is the coordinate;
δ	is the Dirac delta function or heat gradient coefficient $\delta(u)$;
$H(u)$	is the Heaviside unit function;
u	is the mass content (moisture content);
r	is the specific heat of phase transformation;
a_m	is the coefficient of moisture diffusion;
P	is the total pressure;

p_1	is the vapor partial pressure;
p_2	is the inert gas partial pressure;
c_p	is the isobaric specific heat;
k_p	is the coefficient of convective diffusion;
V	is the volume;
$\mathbf{v}$	is the velocity of fluid or gas motion;
Π	is the porosity;
τ, θ	are the time;
λ	is the coefficient of heat conduction;
$\beta(\theta)$	is the internal energy relaxation function in (1.9);
$\alpha(\theta)$	is the internal energy relaxation function in (1.13);
$\lambda(0)$	is the instantaneous coefficient of heat conduction;
$\xi(\tau)$	is the coordinate of the phase transformation surface;
ρ	is the density;
ε	is the phase transformation criterion;
ψ	is the heat and mass transport potential.

Subscripts

1, 2	are the zones separating the body, respectively, in Stefan problems (formulas (2.1)-(2.9));
$i (i = 0, 1, 2, 3)$	are the states of the substance in a capillary-porous body (0 is the body skeleton, 1 is the vapor, 2 is the fluid, and 3 is the inert gas in a porous body);
$i, j = 1, 2, 3$	are the subscripts of the kinetic transport coefficients (3.1);
f	is the phase transformation surface;
m	is the maximum value;
e	is the equivalent value;
r	is the relaxation value.

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